A characterization of Whitney a-regular complex analytic stratifications

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Abstract

We give a characterization of Whitney a-regular complex analytic stratifications of complex analytic subvarieties of a complex manifold N which satisfies the Oka property, in terms of the topology on the set of holomorphic maps, between Stein manifolds and N, which are transverse to the stratification. Our result can be seen as a holomorphic version of Trotman's theorem in the real case which says that a-regularity is necessary and sufficient for the openness of sets of maps transverse to a stratification.

1. Introduction

Every complex analytic subvariety of a complex manifold can be stratified into complex submanifolds in such a way that the strata fit together in a nice way (Whitney [17]). In this paper we deal with the problem of characterizing Whitney a-regular complex analytic stratifications of complex analytic subvarieties of a complex manifold N in terms of the topology on the set of holomorphic maps with N as their target manifold which are transverse to the stratification.

In the real case, such a characterization of smooth Whitney a-regular stratifications was achieved by Trotman in [16], where the author proved that a-regularity is necessary and sufficient for the openness of the set of maps transverse to a stratification in the strong topology. In an earlier paper (Trivedi [15]), we showed that such a characterization can be achieved by the openness of a rather huge set of maps for the weak topology. In this paper we show that it is possible to characterize a-regular complex analytic stratifications in terms of smaller open sets of the set of holomorphic maps between complex manifolds, given some conditions on the target manifold.

We begin with the definitions of some important notions and then summarize some known results in the real case. We then mention some recent results in the case of complex manifolds and prove that a-regularity is redundant for the denseness of transverse holomorphic maps (proposition 3.10) contrary to the assumption in a transversality theorem of Forstnerič (theorem 4.3, [3]). We then show that a-regularity is necessary and sufficient for the openness of the set of holomorphic maps (proposition 3.11) from a Stein manifold which are transverse to a stratified subvariety of a complex manifold which satisfies Gromov's ellipticity condition Ell_1 (page 71 in Gromov [7]). This result can be seen as a holomorphic version of Trotman's theorem. Proposition 4.6 in Forstneric [3] says that those complex manifolds which satisfy the Oka property also satisfy Gromov's ellipticity condition Ell_1, this results gives us theorem 3.14 and allows us to give examples, which are given at the end of the paper, illustrating the validity of our results. An analogue of our result in the case of algebraic manifolds (theorem 3.15) is also given.

2010 Mathematics Subject Classification Primary 58A35, 58C10; Secondary 32H02, 32Q28, 32S60
Keywords: Transversality, Stratifications, Weak topology, Complex manifolds, Oka property, Ellipticity.

2. Real case

Let M and N be smooth manifolds. On the set of all smooth maps between M and N, $C^\infty(M,N)$, two topologies can be defined: the weak topology (compact-open topology) where the subbasic neighbourhoods contain those maps which are close, along with their derivatives, on a compact set, and the strong topology (Whitney topology) where basic neighbourhoods contain those maps which are close on a family of compact sets covering M, see page 35 in Hirsch [9] for details. Denote by $C^\infty_W(M,N)$ and $C^\infty_S(M,N)$, the set of all smooth maps between M and N with the weak topology and the strong topology respectively. Notice that an open subset in the weak topology is also open in the strong topology and a dense subset in the strong topology is also dense in the weak topology.

Definition 2.1. A smooth map $f : M \to N$ is transverse to a submanifold $S \subset N$ at $x \in M$, denoted $f \pitchfork_x S$, if either $f(x) \notin S$ or $f(x) \in S$ and $T_{f(x)}S + Df_x(T_xM) = T_{f(x)}N$. And f is transverse to S on $K \subset M$, denoted $f \pitchfork_K S$, if it is transverse at all $x \in K$.

Notice that if the codimension of S is greater than the dimension of M then a map $f : M \to N$ is transverse to S if and only if $f(M) \cap S = \emptyset$, i.e., if the image of M under f is disjoint from S.

Definition 2.2. A stratification Σ of a subset V of a manifold M is a locally finite partition of V into submanifolds of M. The submanifolds in the partition are called strata.

Definition 2.3. Let S_1 and S_2 be two strata of Σ, S_2 is said to be a-regular over S_1 at $x \in S_1 \cap \overline{S_2}$ if for every sequence of points $\{y_i\}$ in S_2 converging to x such that $\lim_{i\to\infty} T_{y_i}S_2$ exists, we have

$$\lim_{i\to\infty} T_{y_i}S_2 = \tau \quad \Rightarrow \quad T_xS_1 \subset \tau.$$

A stratification is called a-regular if for every pair of strata (S_i, S_j), S_j is a-regular over S_i at every point in the intersection $S_i \cap \overline{S_j}$ and S_i is a-regular over S_j at every point in the intersection $S_j \cap \overline{S_i}$ and in this case we say the pair (S_i, S_j) is a-regular.

Definition 2.4. A map $f : M \to N$ is transverse to a stratification Σ of $V \subset N$ at $x \in M$, denoted $f \pitchfork_x \Sigma$, if it is transverse to every stratum in Σ at x. f is transverse to Σ on $K \subset M$, denoted $f \pitchfork_K \Sigma$, if it is transverse to Σ at all $x \in K$.

The celebrated Thom transversality theorem is the following (see Hirsch [9] and Golubitsky and Guillemin [6] for detailed proofs):

Theorem 2.5. *Let M and N be smooth manifolds, $S \subset N$ a submanifold. Then,*

(a) $T_S = \{f \in C^\infty(M,N) : f \pitchfork S\}$ is a dense subset of $C^\infty_W(M,N)$ as well as of $C^\infty_S(M,N)$.

(b) Suppose S is closed in N and $K \subset M$. Then $\{f \in C^\infty(M,N) : f \pitchfork_K S\}$ is open in $C^\infty_W(M,N)$ if K is compact and open in $C^\infty_S(M,N)$ if K is closed.

Feldman [2] proved the following generalization of the openness result of theorem 2.5 for the strong topology:

Theorem 2.6. *Let M and N be smooth manifolds and let Σ be an a-regular stratification of a closed subset V of N. Then, $T = \{f \in C^\infty(M,N) : f \pitchfork \Sigma\}$ is an open subset of $C^\infty_S(M,N)$.*

Trotman [16] proved a partial converse to theorem 2.6, which is the following:

Theorem 2.7. *Let M and N be smooth manifolds and let Σ be a stratification of a closed subset V of N such that $T = \{f \in C^\infty(M,N) : f \pitchfork \Sigma\}$ is an open subset of $C^\infty_S(M,N)$. Then, Σ is a-regular 'over the strata'* of dimension $\geqslant \dim N - \dim M$.*

For the weak topology we have the following results analogous to theorems 2.6 and 2.7, see Trivedi [15]:

Theorem 2.8. *Let M and N be smooth manifolds and let Σ be an a-regular stratification of a closed subset V of N. Then for any compact set $K \subset M$, the set $T_K = \{f \in C^\infty(M,N) : f \pitchfork_K \Sigma\}$ is an open subset of $C^\infty_W(M,N)$.*

Theorem 2.9. *Let M and N be smooth manifolds and let Σ be a stratification of a closed subset V of N. Let $m \in M$ and suppose that the set $T_m = \{f \in C^\infty(M,N) : f \pitchfork_m \Sigma\}$ is an open subset of $C^\infty_W(M,N)$. Then, Σ is a-regular over the strata of dimension $\geqslant \dim N - \dim M$.*

Once we know the statements of the above theorems, what makes them not very hard to believe is the abundance of smooth maps between smooth manifolds. The existence of bump functions and partition of unity allows us to work locally on the smooth manifolds and then extend the results globally using them. The 'flexibility' of smooth maps allows us to easily perturb a given smooth map to get a transverse map. On the contrary, when we don't have such tools to work with, it is difficult to expect such results to hold, which is the case when we have complex manifolds and holomorphic maps. In fact, the strong topology for holomorphic maps has no meaning (it becomes discrete) and only the weak topology makes sense.

3. Holomorphic case

3.1 Topology on the set of holomorphic maps

Let M and N be complex manifolds. Denote by $\mathcal{H}(M,N)$, the space of all holomorphic maps between M and N with the weak topology. In fact $\mathcal{H}(M,N)$ has a well defined metric and with respect to this metric it is complete. Thus, $\mathcal{H}(M,N)$ is a Baire space. Many interesting sets in $\mathcal{H}(M,N)$ are open. Of particular interest is the set of holomorphic immersions between holomorphic maps. In the following, we will show that the set of holomorphic immersions between M and N on any compact set $K \subset M$ is an open set of $\mathcal{H}(M,N)$.

Let M and N be C^1 manifolds and let $f : M \to N$ be a C^1 map. Let (ϕ, U) and (ψ, V) be coordinate charts in M and N respectively. We define $f_{\phi,\psi} = \psi \circ f \circ \phi^{-1}$.

Lemma 3.1. *Let $f \in C^1(M,N)$ be an immersion at some point $x \in M$. Then, there exist coordinate charts (U,ϕ) around x, (V,ψ) around $f(x)$ with $f(U) \subset V$ and an $\epsilon > 0$, such that for all compact subsets $K \subset U$, every member of $\mathcal{N}(f,(\phi,U),(\psi,V),K,\epsilon)$*† *is an immersion at each point of K.*

Proof. Let (ψ, V) be a chart at $f(x)$ and let (ϕ, U') be a chart at x such that $f(U') \subset V$.

Denote by $L(\mathbb{R}^m, \mathbb{R}^n)$, the set of all linear maps between $\mathbb{R}^m$ and $\mathbb{R}^n$; it is a normed space and has a well defined metric, say δ. Let $\mathcal{I}^c$ be the set of all non injective maps in $L(\mathbb{R}^m, \mathbb{R}^n)$, which is a closed subset.

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*every pair of strata (S_i, S_j) with dimensions $\geqslant \dim N - \dim M$ is a-regular.

†Set of all maps $g \in C^1(M,N)$ such that $g(K) \subset V$, $||f_{\phi,\psi}(x) - g_{\phi,\psi}(x)|| < \epsilon$ and $||Df_{\phi,\psi}(x) - Dg_{\phi,\psi}(x)|| < \epsilon$ for all $x \in \phi(K)$

Define $\eta : U' \to \mathbb{R}$ by $\eta(u) = \delta(D_{\phi(u)} f_{\phi,\psi}, \mathcal{I}^c)$. Then η is a continuous map and since f is an immersion at x, $\eta(x) > 0$. Thus, there exists an open set $U \subset U'$ around x such that $\eta(y) > 0$ for all $y \in U$.

Now, for any compact set $K \subset U$, set $\epsilon = \min\{\eta(y) : y \in K\}$. We claim that the subbasic open neighbourhood of f, $\mathcal{N}(f) = \mathcal{N}(f, (\phi|_U, U), (\psi, V), K, \epsilon/2)$ has the required property. For, if $g \in \mathcal{N}(f)$ and $y \in K$,

$$\begin{aligned}
\delta(D_{\phi(y)} f_{\phi,\psi}, D_{\phi(y)} g_{\phi,\psi}) + \delta(D_{\phi(y)} g_{\phi,\psi}, \mathcal{I}^c) &\geqslant \delta(D_{\phi(y)} f_{\phi,\psi}, \mathcal{I}^c) \\
\delta(D_{\phi(y)} g_{\phi,\psi}, \mathcal{I}^c) &\geqslant \delta(D_{\phi(y)} f_{\phi,\psi}, \mathcal{I}^c) \\
&\qquad -\delta(D_{\phi(y)} f_{\phi,\psi}, D_{\phi(y)} g_{\phi,\psi}) \\
\delta(D_{\phi(y)} g_{\phi,\psi}, \mathcal{I}^c) &\geqslant \epsilon - \epsilon/2 \\
\delta(D_{\phi(y)} g_{\phi,\psi}, \mathcal{I}^c) &\geqslant \epsilon/2 > 0
\end{aligned}$$

Thus, g is an immersion at $y \in K$. □

Proposition 3.2. *Let $Imm_K(M,N)$ be the set of maps between M and N which are immersions at each point of $K \subset M$. Then $Imm_K(M,N)$ is an open subset of $C^1_W(M,N)$ if K is a compact set.*

Proof. Let $f \in Imm_K(M,N)$. To prove that $Imm_K(M,N)$ is open, we show that there exists an open neighbourhood of f which is contained in $Imm_K(M,N)$. Since f is an immersion at each $x \in K$, by lemma 3.1 , for each $x \in K$ there exists a chart U_x with the property that for each compact set $K_x \subset U_x$ there is a neighbourhood $N(f, (\phi_x, U_x), (\psi_x, V_x), K_x, \epsilon_x)$ such that each member of this neighbourhood is an immersion on all of K_x. Since K is compact, we can choose a finite subcollection $\{U_{x_1}, \ldots, U_{x_r}\}$ of the coordinate neighbourhoods $\{U_x\}_{x \in K}$, such that $K \subset \cup_{i=1}^r K_{x_i}$. But then the intersection

$$\cap_{i=1}^r N(f, (\phi_{x_i}, U_{x_i}), (\psi_{x_i}, V_{x_i}), K_{x_i}, \epsilon)$$

($\epsilon = \min\{\epsilon_{x_i}\}$) is an open neighbourhood of f and is contained in $Imm_K(M,N)$, as required. □

Corollary 3.3. *Let M and N be complex manifolds and $Imm_K(M,N)$ be the set of all holomorphic maps between M and N which are immersions at each point of K. Then, $Imm_K(M,N)$ is open in $\mathcal{H}(M,N)$ if K is compact.*

Proof. It follows from the fact that the topology on $\mathcal{H}(M,N)$ is the topology relative to the topology of $C^1_W(M,N)$ considering M and N as C^1 manifolds. □

3.2 Transversality theorems for holomorphic maps

In the case of complex manifolds and holomorphic mappings between them the density result of the Thom transversality theorem is rarely true (see the discussion in Forstnerič [3] about the difficulty in this case.) However, Forstnerič [3] proves that under some conditions on the target manifold it is still possible to prove density results (see also Kaliman and Zaidenberg [10]) and we briefly explain the transversality theorems of Forstnerič [3].

For $x \in M$, $t \in \mathbb{C}^n$ (for some n) and for a holomorphic map $F : M \times \mathbb{C}^n \to N$ define $f_t : M \to N$ by $f_t(x) = F(x,t)$ and $f^x : \mathbb{C}^n \to N$ by $f^x(t) = F(x,t)$.

Definition 3.4. A complex manifold N satisfies condition Ell_1 if for every Stein manifold M and every holomorphic map $f : M \to N$ there exist an $n \geqslant \dim N$ and a holomorphic map $F : M \times \mathbb{C}^n \to N$ such that $f_0 = f$ and f^x is a submersion at $0 \in \mathbb{C}^n$ for each $x \in M$.

Definition 3.5. A complex manifold N satisfies the Oka property if for every Stein manifold M, every compact $\mathcal{H}(M)$-convex subset K of M and every continuous map $f_0 : M \to N$ which is holomorphic on an open neighborhood of K, there exists a homotopy of continuous maps $f_t : M \to N$ such that for every $t \in [0,1]$ the map f_t is holomorphic on a neighbourhood of K and uniformly close to f_0 on K, and the map $f_1 : M \to N$ is holomorphic.

Here $\mathcal{H}(M)$ denotes the sheaf of holomorphic functions on M and a subset U is $\mathcal{H}(M)$-convex means that for every K compact in U, $\hat{K} \subset U$ where $\hat{K} = \{x \in X : |f(x)| \leqslant \sup_K |f| \ \forall f \in \mathcal{H}(M)\}$.

Definition 3.4 originated from the work of Gromov [7] while definition 3.5 is a classical property and has been studied by several mathematicians, see for example [8] and [5]. Also, see Forstnerič [4] for a detailed account of the development of the research in this field.

We now state the transversality theorem of Forstnerič [3]:

Theorem 3.6. *Let M be a Stein manifold and N be a complex manifold satisfying the Oka property. Let Σ be an a-regular stratification of a complex analytic subvariety V of N. Then, the set $T = \{f \in \mathcal{H}(M,N) : f \pitchfork \Sigma\}$ is dense in $\mathcal{H}(M,N)$.*

In the course of proving theorem 3.6 Forstnerič also proves the following proposition:

Proposition 3.7. *Let M and N be complex manifolds and Σ be an a-regular stratification of a complex analytic subvariety V of N. Then, for any compact set $K \subset M$ the set $T_K = \{f \in \mathcal{H}(M,N) : f \pitchfork_K \Sigma\}$ is open in $\mathcal{H}(M,N)$.*

Corollary 3.8. *Let M and N be complex manifolds and S be a complex submanifold of N. Then, for any compact set $K \subset M$ and any compact coordinate disk D of S, the set $T_K = \{f \in \mathcal{H}(M,N) : f \pitchfork_K (S \cap D)\}$ is open in $\mathcal{H}(M,N)$.*

Proof. Take Σ to be the stratification of S having S as the only stratum. Now, the result follows from the fact that the topology on $\mathcal{H}(M,N)$ is the topology relative to the topology of $C^1_W(M,N)$ considering M and N as C^1 manifolds. The openness in the real case of this set is a standard argument as given in [9] or [6]. □

Remark 3.9. Forstnerič uses the term 'Whitney stratification' for a-regular stratifications, however historically the term Whitney stratification is used for a stratification which satisfies b-regularity.

The proof of theorem 3.6 in [3] suggests that it is a necessary condition that the stratification be a-regular. However, we will show that a-regularity is not necessary to prove the denseness of transverse maps.

Proposition 3.10. *Let M be a Stein manifold and N be a complex manifold satisfying the Oka property. Let Σ be a stratification of a complex analytic subvariety V of N. Then, the set $T = \{f \in \mathcal{H}(M,N) : f \pitchfork \Sigma\}$ is dense in $\mathcal{H}(M,N)$.*

Proof. Let $\{B_\alpha\}_{\alpha\in\Lambda}$ denote the strata of Σ. Cover each strata B_α by countably many compact coordinate disks $S^\alpha_k \subset B_\alpha$ in the submanifold B_α for $\alpha \in \Lambda$. Thus,

$$T = \cap_{\alpha\in\Lambda} \cap_{k=1}^{\infty} \{f \in \mathcal{H}(M,N) : f \pitchfork S^\alpha_k\}.$$

Now cover M by countably many compact sets K_j and notice that

$$T = \cap_{\alpha\in\Lambda} \cap_{k=1}^{\infty} \cap_{j=1}^{\infty} \{f \in \mathcal{H}(M,N) : f \pitchfork_{K_j} S^\alpha_k\}. \tag{3.1}$$

By corollary 3.8, for every j, the set on the right hand side of (3.1) is open and it is also dense because it contains the set of all holomorphic maps transverse to Σ on K_j (which is dense in $\mathcal{H}(M,N)$ by Forstnerič's argument). Since $\mathcal{H}(M,N)$ is a Baire space and Σ is a locally finite stratification, T is also dense in $\mathcal{H}(M,N)$. $\square$

Next, we prove a partial converse of proposition 3.7, which is the following:

Proposition 3.11. *Let M be a Stein manifold (dimension m) and N be a complex manifold (dimension n) satisfying the Ell_1 condition. Let Σ be a stratification of a complex analytic subvariety of N, $r = \min\{\dim S : S$ is a stratum in $\Sigma\}$ and $m \geqslant n - r$. Let $K \subset M$ be a compact set[‡] Also T_K is dense in $\mathcal{H}(M,N)$. in M such that the set $T_K = \{f \in \mathcal{H}(M,N) : f \pitchfork_K \Sigma\}$ is open in $\mathcal{H}(M,N)$. Then, Σ is a-regular.*

Proof. We may assume that the strata are of dimension $\geqslant 1$ because a-regularity is automatic over a 0-dimensional stratum. Let $w \in K \subset M$. Since M is Stein, we can assume that M is a complex submanifold of some $\mathbb{C}^p$ such that $w = 0 \in \mathbb{C}^p$, and clearly $p \geqslant n - r$.

Let X and Y be two distinct strata in Σ such that Y is not a-regular over X at some $x \in X \cap \overline{Y}$. Then, there exists a sequence $\{y_i\}_{i=1}^{\infty}$ in Y such that $\lim_{i\to\infty} y_i = x$, $\lim_{i\to\infty} T_{y_i}Y = \tau$ but $T_xX \not\subset \tau$. In fact, by the curve selection lemma [12], we can choose y_i to lie on an analytic curve.

Let $v \in T_xX$ be such that $v \notin \tau$ and E be the one dimensional subspace of T_xN spanned by v. Now, choose a basis for T_xN such that we can write

$$\begin{aligned} T_xX &= E \oplus W_1 \oplus T_1 \\ \tau &= T_1 \oplus T_2 \\ T_xN &= E \oplus W_1 \oplus W_2 \oplus T_1 \oplus T_2 \end{aligned}$$

where T_1, T_2, W_1, W_2 are subspaces of T_xN, $T_1 = T_xX \cap \tau$. Then, find a subspace[§] H of T_xN with $\dim H = \dim N - r$, such that

$$T_2 \oplus W_2 \subseteq H \subseteq T_1 \oplus T_2 \oplus W_1 \oplus W_2.$$

Then, we have

$$H + T_xX = T_xN \tag{3.2}$$

and

$$H + \tau \neq T_xN. \tag{3.3}$$

Let (ψ, V) be a coordinate chart around $x \in N$ such that $\psi(x) = \vec{0} \in \mathbb{C}^n$. Then, $D_x\psi : T_xN \to \mathbb{C}^n$ is a linear isomorphism and under this isomorphism (3.2) and (3.3) become

$$D_x\psi(H) + D_x\psi(T_xX) = \mathbb{C}^n \tag{3.4}$$

and

$$D_x\psi(H) + D_x\psi(\tau) \neq \mathbb{C}^n. \tag{3.5}$$

Choose a basis $\{u_1, \dots, u_m, \dots, u_p\}$ such that $u_1, \dots, u_m$ span the vector subspace T_wM of $\mathbb{C}^p$.

[‡] If K is just a discrete set then the proposition is true even if N does not satisfy the Ell_1 condition, see Trivedi [15].

[§] This trick was first used in the proof of the main theorem in Trotman [16]. See also the remark 3.13.

Let l be the dimension of $D_x\psi(H) + D_x\psi(\tau)$ $(n-r \leqslant l < n)$. Now choose a basis $\{v_1, \dots, v_l\}$ of $D_x\psi(H) + D_x\psi(\tau)$ such that $\{v_1, \dots, v_{n-r}\}$ forms a basis of $D_x\psi(H)$and extend it to a basis $\{v_1, \dots, v_l, v_{l+1}, \dots, v_{n-1}, v'\}$ of $\mathbb{C}^n$ where $D_x\psi(v) = v'$.

Now, define a map $L : \mathbb{C}^p \to \mathbb{C}^n$ (this map is well defined because $p > n - r$) by,

$$L(a_1u_1 + \cdots + a_pu_p) = a_1v_1 + a_2v_2 + \ldots + a_{n-r}v_{n-r}.$$

where $v_1, \dots, v_{n-r}$ is a basis of $D_x\psi(H)$.

Let $\mathscr{D} = \{f \in \mathcal{H}(\mathbb{C}^p, N) : f(w) = x, D_wf(T_wM) = H\}$. Next, we claim that

LEMMA 3.12. *There exists a $g \in \mathscr{D} \cap T_K$*[¶].

Once we have g we can construct a sequence of holomorphic maps $\{g^k\}$ between $\mathbb{C}^p$ and N which converges to g in the weak topology such that for sufficiently large k, $g^k \not\pitchfork_w \Sigma$.

First note that $y_k \in V$ for sufficiently large k. Thus,

$$\lim_{k\to\infty} D_{y_k}\psi(T_{y_k}Y) = D_x\psi(\tau).$$

Now, choose a basis $\{v_1^k, v_2^k, \dots, v_l^k, v_{l+1}^k, \dots, v_{n-1}^k, v^k\}$ of $D_{y_k}\psi(T_{y_k}N)$ such that $D_{y_k}\psi(T_{y_k}Y)$ belongs to the span of $\{v_1^k, \dots, v_l^k\}$ and

$$\lim_{k\to\infty} {v_i}^k = v_i \text{ and } \lim_{k\to\infty} v^k = v. \tag{3.6}$$

Let H^k be the subspace of $D_{y_k}\psi(T_{y_k}N)$ spanned by $v_1^k, \dots, v_{n-r}^k$. Then, by (3.6) we have $\lim_{k\to\infty} H^k = D_x\psi(H)$ and we have

$$H^k + D_{y_k}\psi(T_{y_k}Y) \neq \mathbb{C}^n, \tag{3.7}$$

since the left hand side of (3.7) is spanned by a subset of $\{v_1, \dots, v_l\}$ and $l < n$. Now, define $L^k : \mathbb{C}^p \to \mathbb{C}^n$ by the formula

$$L^k(a_1u_1 + \cdots + a_pu_p) = \psi(y_k) + \psi \circ g(a_1u_1 + \cdots + a_pu_p) + \sum_{i=1}^{n-r} a_i(v_i^k - v_i)$$

and set $g^k = \psi^{-1} \circ L^k$. Clearly then, $g^k(w) = y_k$, by (3.6) the sequence of maps g^k converges to the map g, $D_wL^k(T_wM) = H^k$ and by (3.7), $g^k \not\pitchfork_w \Sigma$.

Now, let $g|_M = f$ and $g^k|_M = f^k$, then clearly f and the f^k's are holomorphic maps between M and N, the sequence f^k converges to f in the weak topology and $f \pitchfork_K \Sigma$ but for large enough k, $f^k \not\pitchfork_w \Sigma$, which is a contradiction to the hypothesis that the set T_K is open in the weak topology. $\square$

Proof of lemma 3.12. We will prove a better result, namely, we show that g can be chosen to be transverse to Σ at every point of M. Notice that $h = \psi^{-1}L \in \mathscr{D}$ and moreover it is a holomorphic immersion on each point of K. Denote $E_K = \{f \in \mathcal{H}(\mathbb{C}^p, N) : f|_K \text{ is an immersion}\}$. By corollary 3.3, E_K is an open set of $\mathcal{H}(\mathbb{C}^p, N)$. Let d be a metric on $\mathcal{H}(\mathbb{C}^p, N)$. Then, there exists a $\delta > 0$ such that $B_\delta(h) = \{f \in \mathcal{H}(M, N) : d(h, f) < \delta\}$ is a subset of E_K. Set $\mathscr{E}_K = \overline{B_{\delta/2}(h)} \cap \mathscr{D}$ (this set is not empty because $h \in \mathscr{E}_K$) and moreover it is a closed subset of $\mathcal{H}(M, N)$, since any converging sequence in $\mathscr{E}_K$ will converge to a point in $\mathscr{E}_K$ as all maps of the sequence are immersions on K. Since closed subsets of complete metric spaces are complete metric, we deduce that $\mathscr{E}_K$ is a Baire space.

[¶]We will in fact show that there exists a map $g \in \mathscr{D} \cap T_K$ such that $g \pitchfork \Sigma$.

Using the above result we can now show that the set $\{f \in \mathscr{E}_K : f \pitchfork \Sigma\}$ is dense in $\mathscr{E}_K$. Let K' be any compact set in M and let $f : \mathbb{C}^p \to N$ be a holomorphic map in $\mathscr{E}_K$. Since N satisfies condition Ell_1, there exists a map $F : \mathbb{C}^p \times \mathbb{C}^{N'} \to N$ as in the definition 3.4. Let $\pi : \mathbb{C}^p \times \mathbb{C}^{N'} \to \mathbb{C}^{N'}$ be the natural projection. Since $f^z : \mathbb{C}^{N'} \to N$ is a submersion for all $z \in M$, there exist a small ball $D \subset \mathbb{C}^{N'}$ around the origin and an open set $U \subset X$ containing K such that F is a submersion of $V = U \times D$ to N. Denote the strata in Σ by $\{B_\beta\}$ and the underlying subvariety by B. Hence $B' = F^{-1}(B) \cap V$ is a complex analytic subvariety of V. Then B' can be stratified by pulling back the strata of B under F (we denote its strata $F^{-1}(B_\beta)$, by B'_β).

Now, we show that if $t \in \mathbb{C}^{N'}$ is a regular value of the restricted projection $\pi : B'_\beta \to D$ then $f_t \pitchfork_z B_\beta$ if $(z,t) \in B'_\beta$.

If $(z,t) \in B'_\beta$ then $y = f_t(z) \in B_\beta$. Since $F(z,t) = y$ and F is a submersion, we know that

$$D_{(z,t)}F(T_{(z,t)}\mathbb{C}^p \times \mathbb{C}^{N'}) + T_y B_\beta = T_y N$$

that is, given any vector $a \in T_y N$, there is a vector $b \in T_{(z,t)}\mathbb{C}^p \times \mathbb{C}^{N'}$ such that

$$D_{z,t}F(b) - a \in T_y B_\beta.$$

We want to exhibit a vector $v \in T_z\mathbb{C}^p$ such that $D_z f_t(v) - a \in T_y B_\beta$. Now,

$$T_{(z,t)}\mathbb{C}^p \times \mathbb{C}^{N'} = T_z\mathbb{C}^p \times T_t\mathbb{C}^{N'},$$

so, $b = (w,e)$ for vectors $w \in T_z\mathbb{C}^p$ and $e \in T_t\mathbb{C}^{N'}$. If e were zero we would be done, for since the restriction of F to $\mathbb{C}^p \times \{t\}$ is f_t, it follows that

$$D_{(z,t)}F(w,0) = D_z f_t(w).$$

If e is not zero, we may use the projection π to kill it off. As

$$D_{(z,t)}\pi : T_z\mathbb{C}^p \times T_t\mathbb{C}^{N'} \to T_t\mathbb{C}^{N'}.$$

is just the projection onto the second factor, and the fact that t is a regular value of restricted projection $\pi : B'_\beta \to D$ we know that there is some vector of the form (u,e) in $T_{(z,t)}B'_\beta$. But, F maps B'_β to B_β, so $D_{(z,t)}F(u,e) \in T_y N$. Consequently, the vector $v = w - u \in T_z\mathbb{C}^p$ is our solution, for

$$D_z f_t(v) - a = D_{(z,t)}F((w,e) - (u,e)) - a = (D_{(z,t)}F(w,e) - a) - D_{(z,t)}F(u,e),$$

and both of the latter vectors belong to $T_y N$.

By Sard's theorem [14], we see that the set of regular values of π is dense in D. Choosing t in this dense set and close to 0 we get maps $f_t : \mathbb{C}^p \to N$ lying in $\mathscr{E}_k$ which are transverse to Σ on K' (in fact on U) and which approximate f on K'.

Thus, the set $\{f \in \mathscr{E}_K : f \pitchfork_{K'} \Sigma\}$ is dense in $\mathscr{E}_K$. This implies that for any stratum B_α of Σ and any compact coordinate disk $K_\alpha \subset B_\alpha$ the set $\{f \in \mathscr{E}_K : f \pitchfork_{K'} B_\beta \cap K_\alpha\}$ is dense in $\mathscr{E}_K$ and corollary 3.8 implies that it is also open. Thus, using the fact that $\mathscr{E}_K$ is a Baire space, we conclude that the set $\{f \in \mathscr{E}_K : f \pitchfork \Sigma\}$ is dense in $\mathscr{E}_K$.

This proves the existence of a $g \in \{f \in \mathscr{E}_K : f \pitchfork \Sigma\} \subset \mathscr{D} \cap T_K$. □

Remark 3.13. The above proof is inspired by the proof of the main theorem in Trotman [16] where the author only mentions the existence of the required maps (in the real case), while we have constructed them explicitly in the holomorphic case.

Combining our proposition 3.11 with the proposition 4.6 of Forstnerič [3], which gives criteria for complex manifolds to satisfy the Ell_1 condition, we have the following theorem:

Theorem 3.14. *Let M be a Stein manifold and N be a complex manifold satisfying the Oka property. Let Σ be a stratification of a complex analytic subvariety in N. Let $K \subset M$ be a compact set in M such that the set $T_K = \{f \in \mathcal{H}(M,N) : f \pitchfork_K \Sigma\}$ is open in $\mathcal{H}(M,N)$. Then, Σ is a-regular over the strata of dimensions $\geqslant \dim N - \dim M$.*

In the case of algebraic maps between algebraic manifolds and algebraically sub elliptic manifolds studied by Forstnerič [3], a similar kind of result can be obtained. We have the following result (we give the statement without proof because the proof is similar to the holomorphic case):

Theorem 3.15. *Let M be an algebraic manifold and N be a subelliptic algebraic manifold (i.e. N satisfies the Ell_1-condition). Let Σ be a stratification of a complex analytic subvariety of N. Let $K \subset M$ be a compact set such that the set $T_K = \{f \in \mathcal{O}(M,N) : f \pitchfork_K \Sigma\}$ is open in $\mathcal{O}(M,N)$ (set of all algebraic maps between M and N). Then, Σ is a-regular over the strata of dimensions $\geqslant \dim N - \dim M$.*

4. Examples

We give two examples where the assumptions of our theorem are not satisfied and our result does not hold.

1. Let M be a compact complex manifold. A compact complex manifold is not Stein and any holomorphic map from M to $\mathbb{C}^n$ is a constant. Let V be a complex analytic subvariety in $\mathbb{C}^n$ and Σ a stratification of V. Then, a holomorphic map $g_x : M \to \mathbb{C}^n$ which maps all points of M to $x \in \mathbb{C}^n$ is transverse to Σ, at any point $m \in M$ if and only if $x \notin V$. Thus, even if Σ is not a-regular, the set of maps which are transverse to Σ is open.

2. A complex manifold N is 'Brody hyperbolic' if there are no non-constant holomorphic maps from $\mathbb{C}$ to N, see [1] or [11]. On the other hand, complex manifolds which satisfy the Oka property are those manifolds which are the target of 'many' non-constant holomorphic maps from $\mathbb{C}^n$. Brody hyperbolic manifolds do not satisfy the Oka property, see the discussion in [11]. Let M be the complex line and N be a polydisk in $\mathbb{C}^n$. The manifold N is a Brody hyperbolic manifold because any holomorphic map from $\mathbb{C}$ to N must be constant by Liouville's theorem. So, once again a map from $\mathbb{C}$ to N is transverse to a stratified set in N if and only if it does not touch the stratified set and so even for non a-regular stratifications the set of maps transverse to them is open. Thus the result does not hold in this case. However, we don't know if our result can be improved for a bigger set of complex manifolds and it needs to be investigated.

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